

Super-connections and non-commutative geometry

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Abstract

We show that Quillen's formalism for computing the Chern character of the index using superconnections extends to arbitrary operators with functional calculus. We thus remove the condition that the operators have, up to homotopy, a gap in the spectrum. This is proved using differential graded algebras and non-commutative differential forms. Our results also give a new proof of the coincidence of the Chern character of a difference bundle defined using super-connections with the classical definition.

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1 Introduction

This paper is a sequel of [27] where the well known McKean-Singer formula was generalized. The problem we are concerned with is to find explicit formulae for the Connes-Karoubi character of the index of an ‘unbounded Fredholm family D parametrized by an algebra B ’. The usual family index theorem corresponds to the case $B = C^\infty(Y)$ where Y is the parameter space.

Given such an unbounded Fredholm family D of hermitian operators on a Hilbert space, its K -theoretical index is defined using the graph projection of D :

$$p = \begin{bmatrix} 1 - e^{-P^*P} & \tau(P^*P)P^* \\ \tau(PP^*)P & e^{-PP^*} \end{bmatrix} \quad (1)$$

where

$$D = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix} \quad (2)$$

and τ is a smooth even function satisfying $\tau(x)^2x^2 = e^{-x^2}(1 - e^{-x^2})$. This projection is also called the Bott or the Wasserman projection by some authors.

In the case of a single Fredholm operator it is easy to see that the relative dimension of the graph projection and the constant projection e_0

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is the same as the index of P . For families the difference of the K -theory classes $[p] - [e_0] \in K_0(C^\infty(Y)) = K^0(Y)$ coincides with the the index bundle defined by Atiyah-Singer [3].

This provides us with a method of computing the Connes-Karoubi classes of the index in cyclic homology. Computations using the graph projections were carried out in [13, 24]. Quite often however it is easier to work with heat kernels rather than the above projection as in [2, 6, 15, 22]. For the families index theorem [3] this requires superconnections [28]. As for connections, to a superconnection there is associated the supercurvature endomorphism

which is a closed form, nonhomogeneous in general. The main fact is then that the Chern character of the index coincides with (the cohomology class of) the exponential of the supercurvature

$$ch(\text{Ind}(D)) = \exp(-(tD + \nabla)^2).$$

The proof of this in the case of families parametrized by a *compact* manifold, goes as follows. One first establishes this principle for operators with gaps in the spectrum and the same index bundle. Then, using the invariance under homotopy of the cohomology class of the exponential of the supercurvature, the general case is reduced to the above mentioned one. This procedure is due to Bismut [6] and was also used in [22].

There are situations, however, where there are no operators with gaps in the spectrum, but superconnections and the exponential of the supercurvature still make sense. A natural problem is what they represent. Situation of this sort appear in the study of foliations, see [18], for families of b-pseudodifferential operators [8, 23], or on open manifolds.

The main result of this paper establishes the equality of the Connes-Karoubi character of the graph projection and of the Chern character of the superconnection whenever some sort of pseudodifferential calculus exists. See Theorem 4.12. The assumptions of our theorem are verified in the classical cases, for pseudodifferential operators along the leaves of a foliation and for open manifolds. It also applies to algebraic (or formal) settings, such as the case of formal deformations $B = C_c^\infty(Y)[[h]]$ of certain commutative algebras.

The main ingredients of the proof are to construct a certain completion of the universal differential algebra of $C_c^\infty(\mathbb{R})$ and to prove the theorem in this formal but universal setting. The proof relies heavily on cyclic cohomology. It is interesting to mention that there are classical cases when our results can be formulated without any reference to noncommutative geometry, but for which no classical proof is available.

Let us mention the following example. Let X be an open σ -compact manifold. The K -theory groups of X with compact support can be defined in two equivalent ways, either as the kernel of $K^*(X \cup \{\infty\}) \rightarrow K^*(\{\infty\})$, or as equivalence classes of triples (E_0, E_1, P) where $P \in \text{End}(E_0, E_1)$ is invertible outside a compact set. We can think of the difference bundle $[E_0] - [E_1] \in K^0(X)$ as “the index of P ”. We can assume that E_0 and E_1 are endowed with hermitian metrics and that $\|P\| \rightarrow \infty$ at ∞ . To such a triple

(E_0, E_1, P) there is associated an element $[p] - [e_0] \in \ker(K^*(X \cup \{\infty\}) \rightarrow K^*(\{\infty\}))$ where p and e_0 are the projections defined above (see definition 4.2). Our main result implies that the super-connection Chern character as defined by Quillen [28] coincides with the classical Chern character $K^*(X) \subset K^*(X \cup \{\infty\}) \rightarrow H^*(X \cup \{\infty\}) \otimes \mathbb{R}$. This result, implicit in Quillen's work, was first published by Berline and Vergne [5].

Other examples and applications are to the Chern character of Dirac operators on foliated manifolds, leading to the vanishing of certain secondary characteristic classes for foliations with positive scalar curvature along the leaves, see [18, 25].

Our main result extends to the odd case, the proof being the same, word for word. This gives a method to compute the Connes-Karoubi character on K_1 using super-connections. In case $n = 1$ this recovers the main result from [16]. Some applications of this theorem to 1-dimensional foliations will be included elsewhere.

Some very interesting results related to the results in this paper are contained in [14] where *Diff*-invariant structures and a ‘universal’ index theorem are treated in detail. See also [12].

I would like to thank Michèle Vergne for bringing her joint work with Nicole Berline [5] to my attention.

2 Connections and traces

We consider following Connes [11] the construction of cyclic cocycles from algebras endowed with connections and traces that vanish on covariant derivatives (i.e. from what are called below ‘cycles’).

A similar problem was solved by Quillen in the framework of the (b, B) -bicomplex [29]. We review and extend here Connes’s approach to the super-algebra case, carefully keeping track of signs. The terms *graded vector space*, \mathbb{Z}_2 -*graded vector space* and *super vector space* will be used interchangeably.

Unless otherwise mentioned all the spaces considered will be locally convex topological spaces, all algebras will be locally convex algebras with jointly continuous multiplications, and all linear maps will be assumed continuous.

We begin by considering a filtered superalgebra $\Omega = F_0\Omega \supset F_{-1}\Omega \supset \dots$ endowed with an odd graded derivation ∇ satisfying $\nabla(F_{-k}\Omega) \subset F_{-k-1}\Omega$. The \mathbb{Z}_2 -degree on an element $a \in \Omega$ will be denoted by ∂a . Thus we have

that $F_{-k}\Omega$ is \mathbb{Z}_2 -graded vector space and that $\nabla(ab) = (\nabla a)b + (-1)^{\partial a}a\nabla b$. The multiplication $F_{-k}\Omega \otimes F_{-l}\Omega \rightarrow F_{-k-l}\Omega$ is even. We do not assume Ω to have a unit.

Recall that a multiplier (l, r) of Ω consists of a pair of linear maps $l, r : \Omega \rightarrow \Omega$ satisfying $l(ab) = l(a)b$, $r(ab) = ar(b)$ and $al(b) = r(a)b$ for any $a, b \in \Omega$. We shall call ∇ a *connection* if there exists a multiplier $\omega = (l, r)$, called the *curvature*, such that $\nabla^2(a) = l(a) - r(a)$ and $[\nabla, l] = [\nabla, r] = 0$. If Ω has a unit multipliers are in one-to-one correspondence with elements of Ω , so $\omega \in F_{-2}\Omega$, $l(a) = \omega a$ and $r(a) = a\omega$. If moreover the filtration of Ω comes from a grading (i.e. $\Omega = \bigoplus_{k=0}^{\infty} \Omega^k$) and $\nabla(\Omega^k) \subset \Omega^{k+1}$ we have $\omega \in \Omega^2$ and the above definition coincides with the usual one. In the unital case we also have $\nabla^2 = ad_{\omega}$ and $\nabla(\omega) = 0$, see [11]. By a *supertrace* on Ω we shall mean a graded trace with respect to the \mathbb{Z}_2 -degree, that is a map $\tau : \Omega \rightarrow \mathbb{C}$ satisfying $\tau(ab) = (-1)^{\partial a \partial b} \tau(ba)$. A supertrace τ will be called *closed* if $\tau(\nabla a) = 0$ for any $a \in \Omega$.

Definition. 2.1 (Connes) (i) A chain (Ω, ∇, τ) is a filtered superalgebra $\Omega = F_0\Omega \supset F_1\Omega \supset \dots$ with an odd connection ∇ satisfying $\nabla(F_{-k}\Omega) \subset F_{-k-1}\Omega$ and a supertrace $\tau : \Omega \rightarrow \mathbb{C}$.

(ii) A chain (Ω, ∇, τ) will be called exact if $\nabla^2 = 0$.

(iii) A chain for which the trace is closed (i.e. $\tau(\nabla a) = 0 \forall a$) is called a cycle.

Recall also [11] that the universal differential (bi)graded algebra associated to a \mathbb{Z}_2 -graded algebra A can be realized as $\Omega(A) = \bigoplus_{n=0}^{\infty} A^+ \otimes A^{\otimes n}$ where $A^+ = A \oplus \mathbb{C}$ is the algebra with an adjoint unit (even if A already had one) and $d(a_0 \otimes \dots \otimes a_n) = 1 \otimes a_0 \otimes \dots \otimes a_n$. The filtration is $F_{-k}\Omega(A) = \bigoplus_{n=k}^{\infty} A^+ \otimes A^{\otimes n}$. See also [1, 21]. Note that the \mathbb{Z}_2 -degree of a form in $\Omega^n(A)$ may be different from $n \pmod{\mathbb{Z}_2}$ if A is not trivially graded.

Recall [11] that a cyclic cocycle φ of order n on a superalgebra A is a multilinear map $\varphi : A^{\otimes n+1} \rightarrow \mathbb{C}$ satisfying

$$\begin{aligned} \varphi(a_0a_1, \dots, a_n) - \varphi(a_0, a_1a_2, \dots, a_n) + \dots + (-1)^{n-1}\varphi(a_0, \dots, a_{n-1}a_n) \\ + (-1)^{\nu+n}\varphi(a_na_0, a_1, \dots, a_{n-1}) = 0 \end{aligned} \quad (3)$$

and

$$\varphi(a_n, a_0, \dots, a_{n-1}) = (-1)^{\nu+n}\varphi(a_0, a_1, \dots, a_n) \quad (4)$$

where $\nu = \partial a_n(\partial a_0 + \dots + \partial a_{n-1})$.

The first equation can also be written as $\varphi \circ b = 0$ where b is the Hochschild boundary [20], and hence (3) means that φ is a Hochschild cocycle. The second equation is the cyclic property in the graded case [11, 20]. One can see from the definition that a 0-cyclic cocycle on A is simply a supertrace. Our definition is taken from [20] and is different from the one used in [21].

We record for convenience some easy generalizations to the graded case of some computations in [11].

Lemma. 2.2 Denote by $\Phi(\alpha_0 \otimes \alpha_1 \otimes \dots \otimes \alpha_n) = (-1)^\mu \alpha_0 d\alpha_1 \dots d\alpha_n$, then

$$\Phi(b(\alpha_0 \otimes \alpha_1 \otimes \dots \otimes \alpha_n)) = (-1)^{n-1} [\Phi(\alpha_0 \otimes \dots \otimes \alpha_{n-1}), \alpha_n] \quad (5)$$

Here $d^2 = 0$, $\mu = \sum_{j=0}^n (n-j)\partial a_j = \partial a_{n-1} + \partial a_{n-3} + \partial a_{n-5} + \dots$, b is the Hochschild boundary and the commutator is the graded commutator: $[a, a'] = aa' - (-1)^{\partial a \partial a'} a' a$.

Proof. Direct computation:

$$\begin{aligned} & \Phi(b(\alpha_0 \otimes \alpha_1 \otimes \dots \otimes \alpha_n)) = \\ &= (-1)^{(n-1)\partial \alpha_0 + \sum_{j=1}^n (n-j)\partial \alpha_j} \alpha_0 \alpha_1 d\alpha_2 \dots d\alpha_n \\ &+ \sum_{k=1}^{n-1} (-1)^{\sum_0^k (n-j-1)\partial \alpha_j + \sum_{j=k+1}^n (n-j)\partial \alpha_j + k + \partial \alpha_k} \alpha_0 d\alpha_1 \dots d\alpha_{k-1} \alpha_k d\alpha_{k+1} \dots d\alpha_n \\ &+ \sum_{k=1}^{n-1} (-1)^{\sum_{j=0}^k (n-j-1)\partial \alpha_j + \sum_{j=k+1}^n (n-j)\partial \alpha_j + k} \alpha_0 d\alpha_1 \dots d\alpha_k \alpha_{k+1} d\alpha_{k+2} \dots d\alpha_n \\ &+ (-1)^{\nu + (n-1)\partial \alpha_n + \sum_{j=0}^{n-1} (n-j-1)\partial \alpha_j + n} \alpha_n \alpha_0 d\alpha_1 \dots d\alpha_{n-1} \end{aligned}$$

The first term cancels with the first term in the first sum, the second term in the first sum cancels with the first term in the second sum, and so on. The only terms that do not cancel are the last term in the second sum and the last term. The computation then follows. \square

Corollary. 2.3 Let $\Omega = F_0 \Omega \supset F_{-1} \Omega \supset \dots$ be a filtered differential superalgebra, $\tau : \Omega \rightarrow \mathbb{C}$ a supertrace satisfying $\tau(F_{-n-1} \Omega) = 0$ and $\rho : A = \Omega / F_{-1} \Omega \rightarrow \Omega$ be an arbitrary set theoretic lifting. Define μ as in lemma 2.2. Then $\varphi(a_0, a_1, \dots, a_n) = (-1)^\mu \tau(\rho(a_0) d\rho(a_1) \dots d\rho(a_n))$ is a Hochschild cocycle independent on the lifting ρ .

Proof. Define $\tilde{\varphi} : \Omega^{\otimes n+1} \rightarrow \mathbb{C}$, $\tilde{\varphi} = \tau \circ \Phi$. Then the previous lemma shows that $\tilde{\varphi} \circ b = 0$. Moreover if any of $\alpha_0, \dots, \alpha_n$ is in $F_{-1}\Omega$ then $\tilde{\varphi}(\alpha_0, \dots, \alpha_n) = 0$. The lemma then follows. \square

Lemma. 2.4 *If in the above Corollary one also has $\tau(da) = 0$ for any $a \in \Omega$ then φ has the cyclic property (equation (4)).*

Proof. We have by definition

$$\begin{aligned} & \varphi(a_n, a_0, \dots, a_{n-1}) \\ &= (-1)^{\partial a_{n-2} + \partial a_{n-4} + \dots + n\partial a_n} \tau(a_n da_0 da_1 \dots da_{n-1}) \\ &= (-1)^{\partial a_{n-2} + \partial a_{n-4} + \dots + \partial a_n (\partial a_0 + \dots + \partial a_{n-1})} \tau(da_0 da_1 \dots (da_{n-1}) a_n) \\ &= (-1)^{\partial a_{n-2} + \partial a_{n-4} + \dots + \partial a_n (\partial a_0 + \dots + \partial a_{n-1}) + \partial a_0 + \partial a_1 + \dots + \partial a_{n-1} + n} \tau(a_0 da_1 \dots da_n) \\ &= (-1)^{\nu+n} \varphi(a_0, a_1, \dots, a_n) \end{aligned}$$

where ν is as in equation (4). \square

We now review an important construction due to Connes.

Let (Ω, ∇, τ) be a cycle with curvature ω . Define $(\tilde{\Omega}, d, \tilde{\tau})$ by

$$\tilde{\Omega} = \Omega \oplus \Omega X \oplus X\Omega \oplus X\Omega X \tag{6}$$

where X is a formal odd degree symbol and is not to be considered alone (i.e. aX, Xa do make sense, but X does not). The filtration on $\tilde{\Omega}$ is defined by $F_{-k}\tilde{\Omega} = F_{-k}\Omega \oplus F_{-k+1}\Omega X \oplus XF_{-k+1}\Omega \oplus XF_{-k+2}\Omega X$. On $\tilde{\Omega}$ we consider the multiplication $(aX)b = a(Xb) = 0 \quad \forall a, b \in \Omega$, $(aX)(Xb) = a\omega b$, and the differential

$$da = \nabla a + Xa + (-1)^{\partial a} aX \quad \forall a \in \Omega, \quad dX = 0 \tag{7}$$

The trace is $\tilde{\tau}(a_{00} + a_{01}X + Xa_{10} + Xa_{11}X) = \tau(a_{00}) - (-1)^{\partial a_{11}} \tau(\omega a_{11})$.

Theorem. 2.5 (Connes) (i) *Using the above notation $(\tilde{\Omega}, d, \tilde{\tau})$ is an exact cycle. If $\tau(F_{-n-1}\Omega) = 0$ then $\tilde{\tau}(F_{-n-1}\tilde{\Omega}) = 0$. Moreover we have $\Omega = e\tilde{\Omega}e$ and $\nabla(a) = e(da)e$ if $a \in \Omega$ and Ω has a unit e .*

(ii) *Suppose Ω is a filtered algebra endowed with a connection ∇ and curvature ω . Let $\rho : B \rightarrow \Omega$ be a degree preserving algebra morphism. Suppose B has a unit denoted e . Then there exists a morphism $\varphi : e\Omega^*(B)e \rightarrow \Omega$ such that $\nabla(\varphi(b)) = \varphi(e(db)e)$ and $\omega = \varphi(edede)$. \square*

Note that the sign in (7) is different from the one in [11], page 329. This is necessary in order to have $d(aXb) = 0$ and for $\tilde{\tau}$ to be a closed trace, as seen from the computation of $\tilde{\tau}(d(aX))$.

It makes sense to adjoin X to the algebra but then $X^2 \neq \omega$ since $d(X^2) = 0 = \nabla\omega \neq d\omega$. This would not contradict $(aX)X = a\omega$ since the unit e of Ω , if it exists, is never a unit of the bigger algebra: $eXa \neq Xa$ if $a \neq 0$.

Part (ii) is the analog in the connection case of the corresponding result for differential graded algebras and immediately follows from that case using (i). Indeed let $\tilde{\Omega} = \Omega \oplus \Omega X \oplus X\Omega \oplus X\Omega X$ be the differential algebra associated to (Ω, ∇) as in (i). Then by universality we get a morphism $\Omega^*(B) \rightarrow \tilde{\Omega}$ whose restriction $e\Omega^*(B)e \rightarrow \Omega = e\tilde{\Omega}e$ is the morphism φ we are looking for. The last statement follows from $\varphi(edede) = e(\nabla e + Xe + eX)(\nabla e + Xe + eX) = \omega$ since $\nabla e = 0$. This is in agreement with equation (7).

Theorem. 2.6 (Connes) *Let (Ω, d, τ) be an exact cycle, and define $A = \Omega/F_{-1}\Omega$. Suppose that $\tau(F_{-n-1}\Omega) = 0$ and choose an arbitrary set theoretic lifting $\rho : A \rightarrow \Omega$ for $\Omega \rightarrow A$. Then*

$$\varphi(a_0, a_1, \dots, a_n) = (-1)^\mu \tau(\rho(a_0)d\rho(a_1) \dots d\rho(a_n))$$

is a cyclic cocycle on A independent of the choice of ρ . Here $\mu = \partial a_{n-1} + \partial a_{n-3} + \partial a_{n-5} + \dots$ \square

The above theorem gives a (non-commutative) geometric way of defining cyclic cocycles and was considered by Connes for trivially graded algebras in [11]. Together with Theorem 2.5 it provides us with a canonical method of construction cyclic cocycles. The proof is contained in the lemmata above. The extra sign μ is necessary in order to get the Hochschild cocycle property in the \mathbb{Z}_2 -graded case. It is interesting to note that the sign was originally obtained from the formula of the Fedosov product using the bivariant Chern-Connes character [26]. A different formula was considered in [21].

Assuming the trace τ to be even, i.e. $\tau(a) = 0$ if a is odd, we also obtain using [27] an odd cyclic cocycle ψ^τ on the crossed product algebra $A \rtimes \mathbb{Z}_2$ defined by

$$2\psi^\tau(a_0v^{i_0}, a_1v^{i_1}, \dots, a_nv^{i_n}) = \\ = \begin{cases} (-1)^\nu \varphi^\tau(a_0, \dots, a_n) = (-1)^{\mu+\nu} \tau(a_0da_1 \dots da_n) & \text{if } \partial a_0 + \dots + \partial a_n \text{ is even and } i_0 + \dots + i_n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Here $\nu = 0$ if $n = 0$ and $\nu = \sum_{k \leq n} i_k(\partial a_{k+1} + \dots + \partial a_n)$ if $n > 0$ and μ is as in the above lemma. Moreover $v^2 = 1$ is the invertible element implementing the action of \mathbb{Z}_2 : $vav^{-1} = \alpha(a)$. In the notation of [27] we have $\psi^\tau = p_A^* \varphi^\tau$.

Definition. 2.7 Let be a cycle with the property that $\tau(F_{-n-1}\Omega) = 0$. The cyclic cocycle associated to (Ω, ∇, τ) (and n) is by definition the cyclic cocycle $\varphi^\tau : A^{\otimes n+1} \rightarrow \mathbb{C}$ defined using Theorems 2.5 (i) and 2.6.

One of the reasons for working in the setting of filtered algebras rather than that of differential graded algebras, besides the connections with the Fedosov product, is that it is closer to the original idea of Quillen [28] of using higher homogeneous components in the superconnection, in addition to the usual degree one component. This is crucial for the following theorem.

Denote by $HC^n(A)$ the cyclic cohomology groups of a superalgebra A as defined in [11] with the necessary changes required by the grading (see [20]). Also let $S : HC^n(A) \rightarrow HC^{n+2}(A)$ be Connes' periodicity operator [11].

The proof of the following theorem will require the notion of cobordism [11].

Definition. 2.8 (Connes) Two cycles $(\Omega_0, \nabla_0, \tau_0)$ and $(\Omega_1, \nabla_1, \tau_1)$ will be called cobordant if there exists a chain $(\Omega', \nabla', \sigma)$ and a morphism $r : \Omega' \rightarrow \Omega_0 \oplus \Omega_1$ which is compatible with connections and curvatures

$$r(\nabla' a) = (\nabla_0 \oplus \nabla_1)r(a), \quad r(\omega') = \omega_0 \oplus \omega_1 \quad (9)$$

(we implicitly assume that r can be extended to include ω' in its range) and which satisfies Stoke's Theorem

$$\sigma(\nabla' a) = (-\tau_0 \oplus \tau_1)(r(a)) \quad (10)$$

Connes' main result on cobordant cycles is that two such closed cycles give rise to cyclic cocycles φ_0, φ_1 satisfying $S\varphi_0 = S\varphi_1$ [11] Theorem 32. (The actual theorem states that $\varphi_1 - \varphi_0 = B\psi$ for some Hochschild cocycle ψ , which is equivalent to the statement we have given above using Connes' exact sequence, loc. cit.) In the following we shall sketch the easy extension of this result to arbitrary cycles (i.e. not necessarily satisfying $\nabla^2 = 0$).

Lemma. 2.9 (i) The functor $\tilde{\cdot}, \Omega \rightarrow \tilde{\Omega}$, from cycles to closed cycles defined in theorem 2.5 preserves direct sums and is linear in τ .

(ii) Moreover the functor \sim preserves the cobordism between cycles.

Proof. The first part is immediate from the definitions.

In order to prove (ii) we shall use the notation in Definition 2.8 and Theorem 2.5 (i). The compatibility equation (9) is obvious. The Stokes theorem (10) is immediate for $a \in \tilde{\Omega}'$ of the form $\alpha, \alpha X$ or $X\alpha$ with $\alpha \in \Omega'$. Suppose now that $a = X\alpha X$ with $\alpha \in \Omega'$, then we have

$$\begin{aligned}\tilde{\sigma}(d(X\alpha X)) &= \\ &= -(-1)^{\partial\alpha} \sigma(\omega' \nabla' \alpha) \\ &= -(-1)^{\partial\alpha} \sigma(\nabla'(\omega' \alpha)) \\ &= -(-1)^{\partial\alpha} (-\tau_0) \oplus \tau_1(r(\omega' \alpha)) \\ &= -(-1)^{\partial\alpha} (-\tau_0) \oplus \tau_1((\omega_0 \oplus \omega_1)r(\alpha)) \\ &= (-\tilde{\tau}_0) \oplus \tilde{\tau}_1(Xr(\alpha)X)\end{aligned}$$

This proves the lemma. \square

Corollary. 2.10 (Connes) *Suppose the cycles $(\Omega_0, \nabla_0, \tau_0)$ and $(\Omega_1, \nabla_1, \tau_1)$ are cobordant and satisfy $\tau_i(F_{-n-1}\Omega_i) = 0$. Let φ_0 and φ_1 the cyclic cocycles associated to these cycles. Then $S\varphi_1 = S\varphi_0$.* \square

Theorem. 2.11 *Let (Ω, ∇, τ) be a cycle. Choose an arbitrary odd element $\eta \in F_{-1}\Omega$. Define $\nabla_1 = \nabla + ad_\eta$ and φ_0 , respectively φ_1 , to be the cyclic cocycles associated to $(\Omega, \nabla_0 = \nabla, \tau)$, respectively to (Ω, ∇_1, τ) . Then $S\varphi_0 = S\varphi_1$.*

Proof. The proof will follow from the above corollary once we prove that the two cycles in the theorem are cobordant. The required cobordism is given by $(\Omega[0, 1] + \Omega[0, 1]dt, \nabla' = \nabla + dt\frac{\partial}{\partial t} + ad_{t\eta}, \sigma)$ where $\sigma(a + (dt)b) = \int_0^1 dt \tau(b(t))$. The morphism r is the evaluation at the end points. (Note that the physicist's notation for integrals is the correct one in the case of noncommuting variables!) The fact that σ is a trace and the relation $\sigma(\nabla_1(a)) = \tau(a(1) - \tau(a(0))$ are easy computations. \square

3 Topological preliminaries

In this section we shall discuss some aspects related to the topologies on the algebras that we are going to work with. We also prove the single generation for the homology of a certain complex similar to the complexes used in [27].

We shall frequently use results and definitions from [27] and we begin by recalling some of them.

Let \mathcal{D}_ϵ ($\epsilon > 0$) be the subalgebra of $C^\infty(\mathbb{R})$ of those functions satisfying the following exponential decay condition at ∞

$$p_{k,\epsilon}(f)^2 = \int_{\mathbb{R}} e^{\epsilon x^2} |f^{(k)}(x)|^2 dx < \infty \quad (\forall) k \geq 0$$

endowed with the Frechet topology generated by the above family of seminorms. Then let $\mathcal{D} = \lim_{\rightarrow} \mathcal{D}_{1/n}$ with the corresponding inductive limit topology. This definition of \mathcal{D} coincides with the one in [27] although the \mathcal{D}_ϵ are different. The new definition has the advantage that it easily gives the following result.

Proposition. 3.1 *The spaces \mathcal{D}_ϵ ($\epsilon > 0$) are nuclear, and hence \mathcal{D} is also nuclear.*

Proof. The last part is an immediate consequence of the properties of the inductive limit [30].

Fix $\epsilon > 0$. It is enough to show that for each of the defining seminorms $p = p_{k,\epsilon}$ there exists another seminorm q such that the inclusion of Hilbert spaces $(\mathcal{D}_\epsilon)_q \rightarrow (\mathcal{D}_\epsilon)_p$ is nuclear [30] (recall that “nuclear” means “trace class” when only Hilbert spaces are involved). We can also restrict to the subspace of the functions vanishing on $(-\infty, 0]$. If one takes $q = p_{k+2,\epsilon}$ then the above spaces are naturally isometrically isomorphic to $L^2([0, \infty))$ with the standard Lebesgue measure and the inclusion becomes the integral operator with kernel $K(x, y) = 0$ if $x > y \geq 0$ and $K(x, y) = (-x + y) \exp(\epsilon(x^2 - y^2)/2)$ if $0 \leq x \leq y$. Since this kernel is square integrable the corresponding operator is Schatten-von Neumann (i.e. $Tr(A^*A) < \infty$). The product of two Schatten-von Neumann operators is trace class and hence we get the result if we let $q = p_{k+4,\epsilon}$. \square

Denote by $\Sigma^{(n)}$ the Fréchet space of symbols of order n on \mathbb{R} :

$$\Sigma^{(n)} = \{f : \mathbb{R} \rightarrow \mathbb{C} : \forall k \geq 0 \exists C_k > 0 \quad |f^{(k)}(x)| < C_k(1 + |x|)^{n-k} \quad \forall x \in \mathbb{R}\} \quad (11)$$

The intersection $\Sigma^{(-\infty)} = \cap \Sigma^{(n)}$ is the Schwartz space of rapidly decreasing functions on \mathbb{R} , also denoted by \mathcal{S} , it has a natural Fréchet topology. The union $\Sigma^{(\infty)} = \cup \Sigma^{(n)}$ of all symbols will have the inductive limit topology. We endow \mathcal{S} and $\Sigma^{(\infty)}$ with the grading $\alpha(f)(x) = f(-x)$.

Let $\widehat{\otimes}$ denote the (completed) projective tensor product [17] and consider the spaces

$$\mathcal{F}_k(m_0, \dots, m_n) = \Sigma^{(m_0)} \widehat{\otimes} \dots \widehat{\otimes} \Sigma^{(m_{k-1})} \widehat{\otimes} \mathcal{D}_{1/m_k} \widehat{\otimes} \Sigma^{(m_{k+1})} \widehat{\otimes} \dots \widehat{\otimes} \Sigma^{(m_n)}$$

where $m_0, \dots, m_n \in \mathbb{N}$. Also define

$$\mathcal{F}_k^{(n)} = \lim_{\rightarrow} \mathcal{F}(m_0, m_1, \dots, m_n) \quad (12)$$

where in the inductive limit all m_j go to ∞ . Obviously $\mathcal{F}_k^{(n)}$ contains $\Sigma^{(\infty) \otimes k-1} \otimes \mathcal{D} \otimes \Sigma^{(\infty) \otimes n-k}$. We want to consider the union of all the spaces $\mathcal{F}_k^{(n)}$.

Proposition. 3.2 *The natural map $\mathcal{F}_k^{(n)} \rightarrow C^\infty(\mathbb{R}^{n+1})$ is injective.*

The idea of the proof is to reduce everything to the compact case using the following lemma. Denote by l^1 the Banach space of absolutely summable sequences of complex numbers.

Lemma. 3.3 *Fix $a > 1$ and let $\chi : \mathbb{R} \rightarrow [0, 1]$ be an even smooth function with support in $[-a, a]$ satisfying $\chi = 1$ on $[-1, 1]$. Denote by $\chi_0 = \chi$ and $\chi_n(x) = \chi(x/a^{2n}) - \chi(x/a^{2(n-1)})$ for $n \geq 1$. Then for any m the map $T : \Sigma^{(m)} \rightarrow l^1 \widehat{\otimes} \Sigma^{(m+1)}$,*

$$T(f) = (\chi_0 f, \chi_1 f, \dots, \chi_n f, \dots)$$

is continuous and in particular $f = \sum \chi_n f$ absolutely in $\Sigma^{(m+1)}$ for any $f \in \Sigma^{(m)}$. The same result holds true for $\mathcal{D}_{1/m} \rightarrow l^1 \widehat{\otimes} \mathcal{D}_{1/(m+1)}$.

Proof. Indeed since the multiplication $\Sigma^{(p)} \otimes \Sigma^{(q)} \rightarrow \Sigma^{(p+q)}$ is continuous it is enough to prove that $1 = \sum \chi_n$ absolutely in $\Sigma^{(1)}$. To show this observe first that due to the definition of χ the support of χ_n is contained in $\{|x| \leq a^{2n+1}\}$ for $n \geq 1$. Hence using $\chi_n(x) = \chi_1(x/a^{2(n-1)})$ we get

$$\sup_x (1 + |x|)^{p-1} |\chi_n^{(p)}(x)| \leq C \sup_x |x|^{p-1} |\chi_n^{(p)}(x)| \leq C a^{-2(n-1)} \sup_y |y|^{p-1} |\chi_1^{(p)}(y)|$$

where C is a positive constant independent on n and $y = x/a^{2(n-1)}$. The estimates for $|1 - \sum_0^N \chi_n|$ are obtained using the same technique. \square

We now go back to the proof of the Proposition 3.2

Proof of the proposition. Suppose $f \in \mathcal{F}_k(m_0, \dots, m_n)$ maps to 0 in $C^\infty(\mathbb{R}^{n+1})$. Let, for any multiindex $\alpha = (k_0, \dots, k_n)$, $\chi_\alpha = \chi_{k_0}\chi_{k_1} \dots \chi_{k_n}$. Also let $|\alpha| = \max\{k_0, \dots, k_n\}$ for α as above. Our assumption on f tells us that each of $f\chi_\alpha$ maps to 0 in $C^\infty(\mathbb{R}^{n+1})$. Since both $C_c^\infty(\mathbb{R})$ and $C^\infty(\mathbb{R})$ are nuclear we can replace the projective tensor product by the injective tensor product and the latter preserves the injectiveness. This shows that $f\chi_\alpha = 0$ in $\mathcal{F}_k(m_0, \dots, m_n)$. Since $f = \sum_\alpha f\chi_\alpha$ in $\mathcal{F}_k(m_0 + 1, \dots, m_n + 1)$ thanks to the above lemma we obtain that f maps to 0 in $\mathcal{F}_k(m_0 + 1, \dots, m_n + 1)$. This proves the proposition. \square

Using the proposition we have just proved we may define

$$\mathcal{F}^{(n)} = \sum_k \mathcal{F}_k^{(n)} \quad (13)$$

as a subspace of $C^\infty(\mathbb{R}^{n+1})$.

From the definition it follows that

$$\mathcal{D}^{\widehat{\otimes} n+1} \subset \mathcal{F}^{(n)} \subset C^\infty(\mathbb{R})^{\widehat{\otimes} n+1} = C^\infty(\mathbb{R}^{n+1}).$$

Since $\Sigma^{(p)} \otimes \mathcal{D}_{1/p} \rightarrow \mathcal{D}_{1/(p+1)}$ is continuous the Hochschild differential b maps $\mathcal{F}^{(n)}$ to $\mathcal{F}^{(n-1)}$ and the above inclusions are chain maps.

We now begin the computation of the homology of $(\mathcal{F}^{(n)}, b)$ which will prove useful in the next section.

Recall that this Hochschild boundary is defined using the grading, see equation (3). In addition to this boundary which we will also denote in the following by b_{grad} in order to avoid any confusion, we will also use b_{ev} and b_{odd} . The second formula is that of the Hochschild boundary defined ignoring of the grading (i.e. consider all $\partial a = 0$). The formula for b_{ev} is given by

$$\begin{aligned} b_{ev}(a_0 \otimes \dots \otimes a_n) &= a_0 a_1 \otimes \dots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \dots \otimes a_n + \dots \\ &\quad - (-1)^n a_0 \otimes \dots \otimes a_{n-1} a_n + (-1)^n \alpha(a_n) a_0 \otimes \dots \otimes a_{n-1} \end{aligned}$$

In the above formula α is the grading morphism.

As seen from the definition these differentials differ only in the form of the last term. Moreover they all commute with the action of the grading. Actually the main reason for considering all these differentials is that $b_{grad} = b_{ev}$ on the *even* parts of the complexes, and $b_{grad} = b_{odd}$ on the *odd* parts. See also [27], Proposition 1. These identification will simplify the computation of the b_{grad} homology as in [27].

We will discuss in detail the b_{ev} homology which is actually the result needed in the next section. The proof in the odd case is similar. We obtain that the extra unbounded operators that we introduce do not change the homology which is again singly generated [27].

Denote $\chi_n^{[m]} = H(nx)(\chi^{[m]}(2^{-m-n/m}x) - \chi^{[m]}(2^{-m-(n-1)/m}x))$ where $\chi^{[m]} : \mathbb{R} \rightarrow [0, 1]$ is a smooth function with support in $[-2^{1/(2m)}, 2^{1/(2m)}]$, $\chi^{[m]} = 1$ on $[-1, 1]$, and H is the Heaviside function, $H(x) = 1$ if $x \geq 0$, $H = 0$ otherwise.

Let $\chi_\alpha^{[m]} = \chi_{\alpha_0}^{[m]} \otimes \dots \otimes \chi_{\alpha_n}^{[m]}$ for any multiindex $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{Z}^{n+1}$. Lemma 3.3 shows that $f = \sum_\alpha \chi_\alpha^{[m]} f$ for any $f \in \mathcal{F}^{(n)}$. Let us introduce the submodules

$$F_{i,m}\mathcal{F}^{(n)} = \{f \in \mathcal{F}^{(n)}, f = \sum_{\alpha \in J_{n,i}} \chi_\alpha^{[m]} f_0\}$$

where $J_{n,i} = \{\alpha \in \mathbb{Z}^{n+1}, \text{ at least one of } |\alpha_n + \alpha_0|, |\alpha_0 - \alpha_1|, \dots, |\alpha_{i-1} - \alpha_i| \text{ is } \geq 2\}$. The first $+$ is not a mistake and is related to the twist in the definition of the b_{ev} . Also observe that the complement of $J_{n,n}$ is a finite set.

The following lemma is the analog of Lemma 3.1 in [9] with the changes imposed by the topology. The proof will be similar.

Lemma. 3.4 *The spaces $F_{i,m}\mathcal{F}^{(n)}$ are invariant under b_{ev} , and for any i the complexes $\cup_m F_{i,m}\mathcal{F}^{(n)}$ have vanishing homologies.*

Proof. The invariance follows from $\chi_p^{[m]} \chi_q^{[m]} = 0$ if $|p - q| \geq 2$. Let $\sigma : F_{i,m-2}\mathcal{F}^{(n)} \rightarrow F_{i,m}\mathcal{F}^{(n+1)}$ be given by

$$\sigma(f)(x_0, \dots, x_{n+1}) = \sum_{|p-q| \geq 2} \chi_p^{[m]}(x_i) \chi_q^{[m]}(x_{i+1}) f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$$

The formula $(b_{ev}\sigma + \sigma b_{ev} + (-1)^i)f \in F_{i-1,m-2}\mathcal{F}^{(n)}$ proves the result. \square

Proposition. 3.5 *The b_{ev} homology is singly generated concentrated in dimension 0 and the generator is even.*

Proof. Define $F_\infty \mathcal{F}^{(n)} = \cup_{i,m} F_{i,m} \mathcal{F}^{(n)}$ and observe that it consists of those functions that vanish in a neighborhood of the 0. This shows that the complex $(\mathcal{F}^{(n)}/F_\infty \mathcal{F}^{(n)}, b_{ev})$ has the same homology as $(\mathcal{F}^{(n)}, b_{ev})$. The first complex is “independent of the topology” since it is concentrated at 0 (this is the analog of the localization used in [27]) and the same computation as in the above lemma shows that it also computes the homology of $(C^\infty([-1, 1])^{\widehat{\otimes} n+1}, b_{ev})$ which is concentrated in dimension 0 and is even as shown in [27]. \square

The complex $\mathcal{F}^{(n)}$ inherits also actions of the cyclic group [10]:

$$t(a_0 \otimes \dots \otimes a_n) = (-1)^{n+\nu} a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$$

if $\nu = \partial a_n(\partial a_0 + \dots + \partial a_{n-1})$ and one can define Connes’ cyclic complex $(\mathcal{F}^{(n)}/(1-t)\mathcal{F}^{(n)}, b_{grad})$ [11].

Theorem. 3.6 *The homology of the complex $(\mathcal{F}^{(n)}/(1-t)\mathcal{F}^{(n)}, b_{grad})$ has dimension one in each positive dimension and the generator c_n in dimension n satisfies $\alpha(c_n) = (-1)^n c_n$. In dimension 0 the even part of the homology is again singly generated by c_0 . The generators can be chosen so that $Sc_{n+2} = c_n$.*

Proof. The proof now continues exactly as in [27]. One sets the analog of the (b, B) -bicomplex of Loday and Quillen from which one gets the Connes’ long exact sequence relating the cyclic homology (as defined by factoring $(1-t)$ as above) and Hochschild homology. The result then follows. \square

4 The Chern character of the index

In this section we will be concerned with finding explicit formulae for the characteristic numbers of the index. We have given such formulae in [27]. In this section we show how super-connections provide an other sequence of such formulae. This gives an other proof of one of the basic formulae in [7] Theorem 2.6 page 109, see also [4] Chapter 9.

We endow \mathcal{D} with the grading $\alpha(f)(x) = f(-x)$. Recall the followin definition from [27]:

Definition. 4.1 *A generalized θ -summable Fredholm operator in A is a degree preserving continuous morphism $D : \mathcal{D} \rightarrow A$, where A is a topological super-algebra.*

A simple but crucial observation is that an operator D as in equation (2) determines uniquely a generalized θ -summable Fredholm operator (denoted by the same letter) in the algebra of bounded linear operators on a Hilbert space \mathcal{H} via functional calculus. Moreover, most importantly, the graph projection (defined by equation (1)) can be written down only in terms of D and the grading automorphism v :

$$p = (1 + v)/2 + D(-e^{-x^2})v + D(\tau(x)x) \quad (14)$$

where τ is a smooth even function satisfying $\tau(x)^2 x^2 = e^{-x^2}(1 - e^{-x^2})$ (as in introduction).

Motivated by this we define the index in general

Definition. 4.2 *The index of a θ -summable Fredholm operator $D : \mathcal{D} \rightarrow A$ is by definition*

$$\text{Ind}(D) = [e_0] - [p] \in K_0(A \rtimes_{\alpha} \mathbb{Z}_2) \quad (15)$$

where $e_0 = (1 + v)/2$ and $[e_0], [p]$ denote the class of the idempotents e_0 and p in the corresponding K -theory group.

Note that $\text{Ind}(D)$ is an odd class in $K_0(A \rtimes \mathbb{Z}_2)$ with respect to the ‘dual’ action of \mathbb{Z}_2 on $A \rtimes \mathbb{Z}_2$ given by $\hat{\alpha}(a + bv) = a - bv$.

Consider the space

$$\overline{\Omega} = \prod_{n=0}^{\infty} \overline{\Omega}^n, \quad \overline{\Omega}^0 = \mathcal{F}^{(0)}, \quad \overline{\Omega}^n = \mathbb{C}1 \otimes \mathcal{F}^{(n-1)} \oplus \mathcal{F}^{(n)}, \quad n > 0 \quad (16)$$

with differential given by $d(m) = 1 \otimes m$, $d(1 \otimes m) = 0$. Obviously $d^2 = 0$. We see from the definition and proposition 3.2 that we have the following inclusions

$$\Omega^*(\mathcal{D}) \subset \overline{\Omega} \subset \prod_n \Omega^n(C^\infty(\mathbb{R}))$$

Using Lemma 2.2 and the fact that $\mathcal{F}^{(\cdot)}$ is invariant under the b we see that the multiplication of $\Omega^*(C^\infty(\mathbb{R}))$ restricts to a multiplication of $\overline{\Omega}$. Similarly the left and right representations of $\Sigma^{(\infty)} = \cup \Sigma^{(n)}$ on $\Omega^*(C^\infty(\mathbb{R}))$ restrict to a representations of $\Sigma^{(\infty)}$ on $\overline{\Omega}$ which are always nonunital since $\Sigma^{(\infty)}(\mathbb{C}1 \otimes \mathcal{F}^{(n-1)}) \subset \mathcal{F}^{(n)}$. These representations are determined by the fact that the algebra $\overline{\Omega}$ has as a “dense” subset the linear span of $a_0 da_1 \dots da_n$ and $da_0 da_1 \dots da_n$ where $a_0, \dots, a_n \in \Sigma^{(\infty)}$ and at least one of them is in \mathcal{D} .

We are going to use the concept of a cycle introduced in definition 2.1. (iii).

Definition. 4.3 A finite summable cycle is a cycle (Ω, ∇, τ) together with a morphism $\rho : \mathcal{D} \rightarrow \Omega$ satisfying:

- (i) the natural morphism $\Omega^*(\mathcal{D}) \rightarrow \tilde{\Omega}$ defined by D extends to a morphism $\bar{\rho} : \overline{\Omega} \rightarrow \tilde{\Omega}$, and
- (ii) $\tilde{\tau}$ vanishes on $\bar{\rho}([\Sigma^{(\infty)}, \overline{\Omega}])$.

In the above definition $\overline{\Omega} \supset \Omega^*(\mathcal{D})$ is as defined above, and $\tilde{\Omega} = \Omega \oplus \Omega X \oplus X\Omega \oplus X\Omega X$ and $\tilde{\tau}$ (the extension of τ) are as in Theorem 2.5.

Remark. We also have $\tilde{\tau} \circ \rho([d\Sigma^{(\infty)}, \overline{\Omega}]) = 0$. This is obtained using the formula $[df, g] = d[f, g] - (-1)^{\partial f}[f, dg]$.

Let \mathfrak{B} be a filtered topological superalgebra containing Ω as an ideal so that $F_{-n}\Omega = \Omega \cap F_{-n}\mathfrak{B}$, and the inclusion $\Omega \rightarrow \mathfrak{B}$ and the multiplications $\Omega \times \mathfrak{B} \rightarrow \Omega$ and $\mathfrak{B} \times \Omega \rightarrow \Omega$ are continuous. Also assume that the connection ∇ on Ω extends to a continuous connection, also denoted ∇ , on \mathfrak{B} , and that $F_{-N}\mathfrak{B} = 0$ for some large N . The following Proposition gives a wide class of examples of finite summable cycles.

Proposition. 4.4 Let (Ω, ∇, τ) be a cycle, $D : \mathcal{D} \rightarrow \Omega^*$ be a continuous morphism, and \mathfrak{B} be as above. Also assume that there exists a continuous extension of D to a morphism $\bar{\rho} : \Sigma^{(\infty)} \rightarrow \mathfrak{B}$ and that the trace τ satisfies $\tau([\mathfrak{B}, \Omega]) = 0$. Then (Ω, ∇, τ) is a finite summable cycle.

Proof. Consider

$$D_{\#} : \Sigma^{(j_0)} \otimes \dots \otimes \Sigma^{(j_{k-1})} \otimes \mathcal{D} \otimes \Sigma^{(j_{k+1})} \otimes \dots \otimes \Sigma^{(j_n)} \rightarrow F_{-n}\tilde{\Omega} = \Omega \cap F_{-n}\mathfrak{B}$$

$$D_{\#}(a_0 \otimes \dots \otimes a_n) = \bar{\rho}(a_0)d\bar{\rho}(a_1)\dots d\bar{\rho}(a_n)$$

Our continuity assumptions on D imply that the above map extends to a continuous map $\mathcal{F}^{(n)} \rightarrow F_{-n}\tilde{\Omega}$, where $\mathcal{F}^{(n)}$ is as defined in the previous section, equation (13). \square

The main reason for introducing the algebra $\overline{\Omega}$ and the notion of θ -summable cycles is to make sense of the exponentials of the curvature of

the super-connection $D + \nabla$. This will then allow us to extend Quillen's method [28].

Denote by $\Delta_n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, t_0 + \dots + t_n = 1\}$ the unit simplex in \mathbb{R}^{n+1} and by $t\Delta_n$ its dilation by t . Also denote by e the unit of $\Sigma^{(\infty)}$, such that $e\overline{\Omega}e = \mathcal{F}^{(0)} \oplus \prod_{n=0}^{\infty} \mathcal{F}^{(n)}$ and define $\nabla(\xi) = e(d\xi)e$. We introduce the short hand notation $\exp(-tD^2)$ for $D(f) \in \Omega$ if $f(x) = \exp(-tx^2)$.

The following Lemmata are devoted to make sense in our setting of the well known formal perturbative expansion:

$$\exp(-t(D + s\nabla)^2) = \sum_{n=0}^{\infty} \tilde{\eta}_n(s; t) \quad (17)$$

where

$$\tilde{\eta}_n(s; t) = (-1)^n \int_{t\Delta_n} \exp(-t_0 D^2) a_0 \exp(-t_1 D^2) a_0 \dots a_0 \exp(-t_n D^2) dV$$

$a_0 = s\nabla(D) + s^2\nabla^2$, and dV is the volume element of $t\Delta_n$ normalized such that $\text{Vol}(t\Delta_n) = t^n/n!$.

We shall make sense of the above formal expression for finite summable cycles by defining $\eta_n(s; t) \in \overline{\Omega}$ which satisfies $\overline{D}(\eta_n(s; t)) = \tilde{\eta}_n(s; t)$, where $\overline{D} : \overline{\Omega} \rightarrow \tilde{\Omega}$ is the extension of $D : \Omega^*(\mathcal{D}) \rightarrow \tilde{\Omega}$.

Consider the functions $g_n : (0, \infty) \times [0, \infty)^{n+1} \rightarrow \Omega^*(C^\infty(\mathbb{R}))$, $g_0(s, t_0) = e^{-t_0 x^2}$ and $g_n(s, t_0, \dots, t_n) = e^{-t_0 x^2} a e^{-t_1 x^2} a \dots a e^{-t_n x^2}$ where $x = id$ is the identity function of \mathbb{R} , $a = se(dx)e + s^2 edede = s(e \otimes x + x \otimes e) + s^2(e \otimes e \otimes e) \in e\overline{\Omega}e$. Remember that x is odd and d is a graded derivation. Also it is important to stress that the multiplication is the one induced by $\Omega^*(C^\infty(\mathbb{R}))$. As a remark on notation we make the convention that $e^{-tx^2} = e_0$ for $t = 0$, where e_0 is the identity of $C^\infty(\mathbb{R})$, different from the identity of $\Omega^*(C^\infty(\mathbb{R}))$. This is in agreement with continuity requirements (see Lemma 4.5 below). We shall write e instead of e_0 below.

It follows from the definition that the functions g_n defined above are continuous and satisfy

$$g_{n+1}(s; t_0, \dots, t_{n+1}) = (sg_n(s; t_0, \dots, t_n)dx + s^2 g_n(s; t_0, \dots, t_n)dede)e^{-t_{n+1}x^2} \quad (18)$$

$$= e^{-t_0 x^2} (sdxg_n(s; t_1, \dots, t_{n+1}) + s^2 dedeg_n(s; t_1, \dots, t_{n+1})) \quad (19)$$

Lemma. 4.5 Define for any $t \geq 0$ the function $f_t(x) = e^{-tx^2}$. Then the function $[0, \infty) \ni t \rightarrow f_t \in \Sigma^{(1)}$ is continuous.

Proof. This function is clearly continuous for $t > 0$ so we only need to check the continuity at 0.

The continuity at 0 is equivalent to

$$\limsup_{t \rightarrow 0} \sup_{x \in \mathbb{R}} |1 - e^{-tx^2}|(1 + |x|)^{-1} = 0 \quad (20)$$

and

$$\limsup_{t \rightarrow 0} \sup_{x \in \mathbb{R}} |(e^{-tx^2})^{(n)}|(1 + |x|)^{n-1} = 0, \quad n \geq 1 \quad (21)$$

The first condition (20) is easy so we concentrate on the second one. We have

$$\sup_{|x| \leq t^{-1/2}} |(e^{-tx^2})^{(n)}|(1 + |x|)^{n-1} \leq \sup_{|x| \leq 1} |(e^{-x^2})^{(n)}|t^{n/2}(1 + t^{-1/2})^{n-1} \leq Ct^{1/2}$$

and

$$\begin{aligned} & \sup_{|x| \geq t^{-1/2}} |(e^{-tx^2})^{(n)}|(1 + |x|)^{n-1} \leq \\ & \leq \sup_{|x| \geq t^{-1/2}} (1 + |x|)^{n-1}/|x|^{n-1} \sup_{|x| \geq t^{-1/2}} |(e^{-tx^2})^{(n)}||x|^{n-1} \\ & = t^{1/2}(1 + t^{1/2})^{n-1} \sup_{|x| \geq 1} |(e^{-x^2})^{(n)}||x|^{n-1} \leq C't^{1/2} \end{aligned}$$

where C and C' are constants independent on t . \square

We now give a combinatorial description of the functions g_n .

Denote by S_n the set of subsets K of $\{0, 1, \dots, n\}$ with the property that $0, n \in K$ and $K^c = \{0, 1, \dots, n\} \setminus K$ contains no consecutive integers. Also denote by $\tilde{C}_n = \{(K, \sigma), K \in S_n, \sigma \in \{0, 1\}^n\}$. The string $\sigma = \sigma_0 \sigma_1 \dots \sigma_{n-1}$, $\sigma_i \in \{0, 1\}$ is going to determine some choices.

To any $(K, \sigma) \in \tilde{C}_n$ we are going to associate the continuous function $f_{K, \sigma} : [0, \infty)^{n+1} \rightarrow \Sigma^{(3)} \times \dots \times \Sigma^{(3)}$ ($n+1$ copies) defined by

$$f_{K, \sigma}(t_0, \dots, t_n) = (e^{-t_0 x^2} x^{i_0}, e^{-t_1 x^2} x^{i_1}, \dots, e^{-t_n x^2} x^{i_n}) \quad (22)$$

The powers $i_p \in \{0, 1, 2\}$ are determined by $i_p = j_p + k_p$ where

$$j_p = \begin{cases} 1 & \text{if } p-1, p \in K \text{ and } \sigma_{p-1} = 1 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$

$$k_p = \begin{cases} 1 & \text{if } p, p+1 \in K \text{ and } \sigma_p = 0 \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

We agree that $x^0 = 1$. Denote by $L(K) \subset K$ the set of elements $p \in K$ such that $p+1$ is also in K . If we denote $m(K) = |L(K)|$ then

$$m(K) = 2|K| - n - 2 = i_0 + i_1 + \dots + i_n \quad (25)$$

which is proved by induction. Here and in what follows $|A|$ denotes the number of elements of a finite set A . Note that the continuity follows from Lemma 4.5.

Here is an example. Let $n = 7$, $K = \{0, 1, 3, 4, 5, 6, 7\}$ and $\sigma = 0\sigma_1\sigma_21101$. Then $L(K) = \{0, 3, 4, 5, 6\}$, $i_1 = i_2 = i_3 = i_6 = 0$, $i_0 = i_4 = i_7 = 1$ and $i_5 = 2$:

$$f_{K,\sigma}(t_0, \dots, t_7) = (e^{-t_0x^2}x, e^{-t_1x^2}, e^{-t_2x^2}, e^{-t_3x^2}, e^{-t_4x^2}x, e^{-t_5x^2}x^2, e^{-t_6x^2}, e^{-t_7x^2}x)$$

Note that the above function does not depend upon σ_1 and σ_2 .

We define C_n to be the set of equivalence classes in \tilde{C}_n defined by the equivalence relation \simeq , where $(K, \sigma) \simeq (K, \sigma')$ if and only if $\sigma_p = \sigma'_p$ for any $p \in L(K)$. Let $K \in S_n$, $K \neq \{0\}$, we will denote by $K^p = (K \setminus \{0\}) - 1 \in S_{n-1}$ if $1 \in K$, and $K^p = (K \setminus \{0\}) - 2 \in S_{n-2}$ if $1 \notin K$. Then we define by induction $\nu_{\{0\}} = 0$, $\nu_K = \nu_{K^p} + 1$ if $1 \notin K$, and $\nu_K = \nu_{K^p} + n + 1$ if $1 \in K$. Let

$$|(K, \sigma)| = \sum_{p \in L(K)} \sigma_p + \nu_K$$

which obviously depends only on the equivalence class of (K, σ) .

We are going to make the convention that, unless otherwise specified, *all integrals are integrals of continuous functions with values in the complete locally convex space $\Omega^*(C^\infty(\mathbb{R}))$* . Also we make the following conventions. The sets denoted by K give the position of the exponentials, those denoted by L give which positions are occupied by 1's and the sets denoted by M will identify which positions are either e 's or $e^{-sx^2}x^2$.

Define, using the above notations, $\eta_n(s; t) \in \Omega^*(C^\infty(\mathbb{R}))$ by the equation

$$\eta_n(s; t) = (-1)^n \int_{t\Delta_n} g_n(s, t_0, \dots, t_n) dV(t_0, \dots, t_n) \quad (26)$$

and

$$\chi_{K,\sigma}(t) = (-1)^{|(K,\sigma)|} \int_{t\Delta_K} f_{K,\sigma} dV \quad (27)$$

where $(K, \sigma) \in C_n$, $\Delta_K = \{t \in \Delta_n, \text{supp}(t) \subset K\}$. A consequence of the way $f_{K,\sigma}$ was defined is that it depends only on σ_p with $p \in L(K)$, so by abuse of notation we will write from now on $f_{K,\sigma}$ where $(K, \sigma) \in C_n$ is an equivalence class.

Note that from the definitions the largest element $\max(K) = n$ if $K \in S_n$.

In the statement of the following Lemma we are going to use the function $\Phi : C^\infty(\mathbb{R})^{\otimes n+1} \rightarrow \Omega^*(C^\infty(\mathbb{R}))$ defined in Lemma 2.2.

Lemma. 4.6 *Let $\psi_n(t) = \sum_{(K,\sigma) \in C_n} \chi_{K,\sigma}(t)$ and $\omega_n(t) = \Phi(\psi_n(t))$.*

(i) *The functions $\omega_n(t)$ satisfy $\omega_0(t) = e^{-tx^2}$ and fit into the following recurrence relation*

$$\omega_{n+1}(t) = - \int_0^t e^{-sx^2} (dx\omega_n(t-s) + dede\omega_{n-1}(t-s)) ds, \quad n \geq 0 \quad (28)$$

if we let $\omega_{-1}(t) = 0$.

$$(ii) \sum_{n=0}^{\infty} \eta_n(s; t) = \sum_{n=0}^{\infty} s^n \omega_n(t).$$

$$(iii) \eta_n(s; t) = \sum_{|K|=n+1} s^{\max(K)} \Phi(\chi_{K,\sigma}(t))$$

(iv) *The functions $\omega_n(t)$ also satisfy the recurrence relation*

$$\omega_{n+1}(t) = - \int_0^t (\omega_n(t-s) dx + \omega_{n-1}(t-s) dede) e^{-sx^2} ds \quad (29)$$

Proof. The proof of (i) is obtained by induction as follows.

We get from the definition that $e\omega_n(t)e = \omega_n(t)$ for any $n \geq 0$ and $t \geq 0$. Using these relations, the fact that x is odd and $e(de)e = 0$ we see that the right hand side of the equation (28) can be written as

$$\int_0^t e^{-sx^2} (dx\omega_n(t-s) + dede\omega_{n-1}(t-s)) ds =$$

$$= \int_0^t e^{-sx^2} (xd\omega_n(t-s) + d(x\omega_n(t-s)) + ded\omega_{n-1}(t-s))ds$$

Using the induction hypothesis for $\omega_n(t)$ and $\omega_{n-1}(t)$ we obtain the following relations

$$\begin{aligned} \int_0^t e^{-sx^2} xd\omega_n(t-s)ds &= - \sum_{(K,\sigma) \in C_n} \Phi(\chi_{K',0\sigma}) \\ \int_0^t e^{-sx^2} d(x\omega_n(t-s))ds &= - \sum_{(K,\sigma) \in C_n} \Phi(\chi_{K',1\sigma}) \end{aligned}$$

where $K' = \{0\} \cup (K+1)$

$$\int_0^t e^{-sx^2} ded\omega_{n-1}(t-s))ds = - \sum_{(K,\sigma) \in C_{n-1}} \Phi(\chi_{K'',\sigma''})$$

where $K'' = \{0\} \cup (K+2)$ and $\sigma''_{p+2} = \sigma_p$.

Note that in both cases $(K')^p = K$ and $(K'')^p = K$ and that $L(K') = \{0\} \cup (L(K)+1)$ and $L(K'') = L(K)+2$. The above relations justify the choice of the signs $(-1)^{|(K,\sigma)|}$ in agreement with Lemma 2.2.

Since, using the above notation, C_{n+1} is the disjoint union of the sets $\{(K',0\sigma), (K,\sigma) \in C_n\}$, $\{(K',1\sigma), (K,\sigma) \in C_n\}$ and $\{(K'',\sigma''), (K,\sigma) \in C_{n-1}\}$ the above relations immediately give (i).

Write $\sum_{n=0}^{\infty} \eta_n(s; t) = \sum_{n=0}^{\infty} \omega'_n(s; t)$ where $\omega'_n(s; t)$ is homogeneous of degree n . The equations (19) and (26) show that $s^n \omega_n(t)$ and $\omega'_n(s; t)$ satisfy the same recurrence relation with the same initial terms 0 and e^{-tx^2} so they are equal. This takes care of (ii). The relation (iii) is just a restatement of (ii).

The last recurrence relation holds true because it does so for $\omega'_n(1; t)$ as seen from the equation (18). \square

Lemma 4.7 *We have $\Phi(\chi_{K,\sigma}(t)) \in \mathcal{F}^{(n)}$ for any $(K,\sigma) \in C_n$.*

Proof. Observe that for any fixed k and $\epsilon > 0$ the restriction of the function $\chi_{K,\sigma}(t_0, t_1, \dots, t_n)$ to $t_k \geq \epsilon$ is continuous as a function to $\mathcal{F}_k^{(n)}$ ($\mathcal{F}_k^{(n)}$ is as defined in the previous section, equation (12)). \square

From the above two lemmata we obtain

Corollary. 4.8 We have $\eta_n(s; t), \omega_n(t), \sum_{n \geq 0}^{\infty} \omega_n(t) \in \overline{\Omega}$ $\forall n \geq 0, \forall t > 0$. \square

Definition. 4.9 We define the exponential of the (super-)curvature in $\overline{\Omega}$ by the equation

$$\exp(-t(x + s\nabla)^2) = \sum_{n=0}^{\infty} s^n \omega_n(t) \in \overline{\Omega}$$

The following Proposition justifies the previous definition.

Proposition. 4.10 (i) $\omega_0(t) = e^{-tx^2}$ for any $t \geq 0$ and $\omega_n(0) = 0$ for $n > 0$.
(ii) $\frac{\partial}{\partial t} \sum_{n=0}^{\infty} s^n \omega_n(t) = -(x^2 + se(dx)e + s^2 edede) \sum_{n=0}^{\infty} s^n \omega_n(t)$ for any $t \geq 0$ where the derivative is in $\Omega^*(C^\infty(\mathbb{R}))$.
(iii) $[x, \omega_n(t)] + e(d\omega_{n-1}(t))e = 0$.

Proof. The first part is a direct consequence of the definitions.

The relation in (ii) is equivalent to

$$\frac{\partial}{\partial t} \omega_{n+1}(t) = -x^2 \omega_{n+1}(t) - e(dx)\omega_n(t) - edede\omega_{n-1}(t)$$

which follows by differentiating the equations in Lemma 4.6 (i). (Use the substitution $s \rightarrow t - s$.)

We proceed by induction on n using the result in (ii).

Let $F(t) = [x, \omega_n(t)] + e(d\omega_{n-1}(t))e = 0$. We have using the induction hypothesis that $\frac{\partial}{\partial t} F(t) = -x^2 F(t)$. This shows that the function $t \rightarrow e^{tx^2} F(t)$ has vanishing first derivative so it is constant. The constant is 0 as follows from (i). \square

The simplex $\Delta_{\{-1\} \cup K}$ is defined by $\Delta_{\{-1\} \cup K} = \{(t_{-1}, t_0, \dots, t_n), t_i \geq 0, \sum t_i = 1, \text{supp}(t) \subset \{-1\} \cup K\}$. Let

$$\tilde{\chi}_{K,\sigma}(t) = (-1)^{|(K,\sigma)|} \int_{t\Delta_{\{-1\} \cup K}} e^{-t_{-1}x^2} \otimes h_{K,\sigma} dV$$

Theorem. 4.11 *The functions $\psi_n(t)$ define cycles in Connes' cyclic complex. More precisely we have the following*

- (i) $b(\psi_{n+1}(t)) = -\tilde{\psi}_n(t)$ where $\tilde{\psi}_n(t) = \sum_{(K,\sigma) \in C_{n-1}} \tilde{\chi}_{K,\sigma}(t)$.
- (ii) $\tilde{\psi}_n(t) \in (1 - \mathbf{t}_n)\mathcal{F}^{(n)}$ where $\mathbf{t}_n(a_0 \otimes \dots \otimes a_n) = (-1)^{n+\nu} a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$ defines the action of the cyclic group \mathbb{Z}_{n+1} on \mathcal{F}^n for $\nu = \partial a_n(\partial a_0 + \dots + \partial a_{n-1})$.

We are going to use Lemma 2.2 quite often so some comments are in order.

According to the definition $\Omega^*(C^\infty(\mathbb{R}))$ contains two copies of the projective tensor product $C^\infty(\mathbb{R})^{\otimes n+1}$, one of them is the image of Φ and the other one is the image of $d \circ \Phi$. It is important to point out that Φ is not the identity but introduces some signs according to the definition in Lemma 2.2. Also we are going to use the same symbol Φ for different values of n . This should cause no confusion.

The grading automorphism α on each component of the tensor products gives rise, by means of Φ , to a grading P on $\Omega^*(C^\infty(\mathbb{R}))$ which is different however from the standard grading denoted also α exactly because d is odd. The precise relations are given by $P(\xi) = (-1)^n \alpha(\xi) = (-1)^{n+\partial\xi} \xi$ for $\xi \in \Omega^n(C^\infty(\mathbb{R}))$. Thus dx is even for the α grading but odd for P . Also $\alpha(\omega_n(t)) = \omega_n(t)$ but $P(\omega_n(t)) = (-1)^n \omega_n(t)$.

The reason for introducing P is the following equation

$$\Phi^{-1}(adb) = P(\Phi^{-1}(a)) \otimes b \quad \forall a \in \text{Im}(\Phi) \subset \Omega^*(C^\infty(\mathbb{R})), b \in C^\infty(\mathbb{R}) \quad (30)$$

We are ready now to start the proof.

Proof of the Theorem. Let $\Omega_{n+1}(t) = \omega_n(t)dx + \omega_{n-1}(t)dde$. Lemma 4.6 (iv) gives that

$$-\Phi(\psi_{n+1}(t)) = \int_0^t e^{-sx^2} \Omega_{n+1}(t-s) ds + \int_0^t [\Omega_{n+1}(t-s), e^{-sx^2}] ds$$

Using then Lemma 2.2 we obtain that

$$-\psi_{n+1}(t) = \int_0^t e^{-sx^2} \Phi^{-1}(\Omega_{n+1}(t-s)) ds + b \int_0^t \Phi^{-1}(\Omega_{n+1}(t-s)) \otimes e^{-sx^2} ds$$

Using again Lemma 2.2, the definition of $\Omega_{n+1}(t)$ and equation (30) we obtain

$$-\Phi(b\psi_{n+1}(t)) = (-1)^n \int_0^t e^{-sx^2} ([P(\omega_n(t-s)), x] + [P(\omega_{n-1}(t-s))de, e])ds$$

which finally gives using Proposition 4.10 (iii)

$$\begin{aligned} & (-1)^{n+1}\Phi(b\psi_{n+1}(t)) \\ &= -P \int_0^t e^{-sx^2} ([\omega_n(t-s), x] - \omega_{n-1}(t-s)de)ds \\ &= -P \int_0^t e^{-sx^2} d\omega_{n-1}(t-s)ds \\ &= -P\Phi(\tilde{\psi}_n(t)) = (-1)^n\Phi(\tilde{\psi}_n(t)) \end{aligned}$$

This proves (i). In order to prove (ii) we begin by making some remarks.

Denote for $(K, \sigma) \in C_n$ by $\phi(K, \sigma) = \{p \in \{0, 1, \dots, n\} : i_p = 1\} \subset K$ where i_p are the exponents appearing in the definition of $f_{K, \sigma}$, see equation (22).

Claim 1. $n - |\phi(K, \sigma)|$ is even. Given any subset $L \subset K$ such that $n - |L|$ is even there exists at most one equivalence class $(K, \sigma) \in C_n$ such that $\phi(K, \sigma) = L$. Order increasingly $\{0, 1, \dots, n\} \setminus L = \{p_0, p_1, \dots, p_{2l}\}$, then a solution exists if and only if $M(L) = \{p_0, p_2, p_4, \dots, p_{2l}\} \subset K$.

Proof of the Claim. We know that $i_0 + i_1 + \dots + i_n = 2|K| - n - 2$ (equation 25) from which it immediately follows that the number of 1's has the same parity as n .

Suppose first that $K = \{0, 1, \dots, n\}$. We have the relation

$$i_0 + i_1 + \dots + i_q = q + 1 - \sigma_q, \quad q = 0, 1, \dots, q-1 \quad (31)$$

The above equation shows that $\sigma_q = \sigma_{q-1}$ if $i_q = 1$, that $\sigma_{q-1} = 0$ and $\sigma_q = 1$ if $i_q = 0$, and that $\sigma_{q-1} = 1$ and $\sigma_q = 0$ if $i_q = 2$. We also see that $K \setminus L$ tells us where changes in the string σ occur. From this it follows that, after ignoring the exponents equal to 1, the positions of the exponents equal to 0 alternate with those equal to 2. The above equation tells us that since $1 + 1 + \dots + 1 + 2$ is “too big”, we necessarily have to begin the alternation procedure with a 0. This determines all i_q 's and hence all σ_q 's again in view of the previous equation. This proves the existence and uniqueness in the case $K = \{0, 1, \dots, n\}$.

In order to determine uniqueness and when solutions exist for an arbitrary given K observe that if $(K_1, \sigma) \in C_n$ and $\sigma_{p-1} = 1$ and $\sigma_p = 0$ then $K = K_1 \setminus \{p\}$ is still in S_n and $\phi(K, \sigma) = \phi(K_1, \sigma)$. Thus we can add elements to K if we appropriately define the previously undefined σ 's.

This proves the uniqueness. Moreover to every element that we add to K we get the corresponding exponent to change from 0 to 2 and all the other ones to stay put. This means that in order for the above procedure to allow us to decrease the number of elements in K we can take out only elements that have the corresponding exponent equal to 2. If we examine the existence in the case $K = \{0, 1, \dots, n\}$ we see that the above positions are $p_1, p_3, \dots, p_{2l-1}$.

This proves the claim.

Note for further reference that if (K, σ) and (K_1, σ) are as above then $\nu_K = \nu_{K_1} \pmod{2}$ and hence

$$|(K, \sigma)| = |(K_1, \sigma)| + 1 \pmod{2} \quad (32)$$

Consider three disjoint sets $L, M, M_1 \subset \{-1, 0, 1, \dots, n\}$ such that $n - |L|$ is even and $|M| = (n - |K|)/2$.

We define

$$\zeta_{L,M,M_1}(t_{-1}, t_0, \dots, t_n) = f_{-1}(t_{-1}) \otimes f_0(t_0) \otimes \dots \otimes f_n(t_n)$$

where

$$f_p(s) = \begin{cases} e^{-sx^2} & \text{if } p \in M \\ e^{-sx^2}x & \text{if } p \in L \\ e & \text{if } p \in M_1 \\ e^{-sx^2}x^2 & \text{otherwise} \end{cases}$$

Also consider

$$\tilde{\zeta}_{L,M,M_1}(t) = \int_{t\Delta_S} \zeta_{L,M,M_1} dV$$

where $S = \{-1, 0, 1, \dots, n\} \setminus M_1$.

Claim 2. $\zeta_{L,M} = \sum_{M_1 \subset M} (-1)^{|M_1|} \tilde{\zeta}_{L,M,M_1}$ where the sum is over all subsets of $\{-1, 0, 1, \dots, n\} \setminus (L \cup M)$, does not depend on M .

Proof of the Claim. Since all the sets M have the same number of elements in $\{0, 1, \dots, n\} \setminus L$ it is enough to show that $\zeta_{L,M}(t) = \zeta_{L,M'}(t)$ for M and

M' differing by exactly one element. More precisely we assume that $M = M'' \cup \{j\}$ and $M' = M'' \cup \{k\}$ (the case when M must be empty is trivial).

The difference $\zeta_{L,M}(t) - \zeta_{L,M'}(t)$ can then be written as the sum of all terms of the form

$$(-1)^{|M_1|} (-\tilde{\zeta}_{L,M,M_1 \cup \{k\}} - \tilde{\zeta}_{L,M',M_1 \cup \{j\}} + \tilde{\zeta}_{L,M,M_1} + \tilde{\zeta}_{L,M',M_1})$$

where M_1 ranges through all subsets of $\{-1, 0, 1, \dots, n\} \setminus (L \cup M'' \cup \{k, j\})$. A simple integration with respect to the extra variable shows that the above term is 0. (Use the same argument as in [27], equation (2.5), page 450.)

The proof of the Claim is now complete.

The above claim tells us that we can define $\zeta_L(t) = (-1)^{\Sigma(L)} \zeta_{L,M}(t)$ where $\Sigma(L) = \sum_{k \in L} (n - k)$, for an arbitrary choice of M .

Claim 3. $\tilde{\psi}_n(t) = (-1)^n \sum \zeta_L(t)$ where L runs through all subsets of $\{0, 1, \dots, n\}$ with the property that $n - |L|$ is even.

Proof of the Claim. It follows from the first claim that

$$e^{t_1} \otimes f_{K,\sigma}(t_0, \dots, t_n) = \zeta_{L,M(L) \cup \{-1\}, M_1}(t_{-1}, t_0, \dots, t_n)$$

for $L = \phi(K, \sigma)$ and $M_1 = \{0, 1, \dots, n\} \setminus K$. Moreover $|K, \sigma| = n + \Sigma(L) + |M_1| \pmod{2}$, this follows from equation (32) by induction on $|M_1|$. For $M_1 = \emptyset$ we use $\nu_K = (n+1)(n+2)/2 - 1$ if $K = \{0, 1, \dots, n\}$ and $\sigma_0 + \sigma_1 + \dots + \sigma_{n-1} = \sum k = 0^n(n-k)i_k$. Since the last sum coincides with $\Sigma(L) \pmod{2}$ this shows that in the next two sums all the terms coincide:

$$\sum_{(K,\sigma), \phi(K,\sigma)=L} \tilde{\chi}_{K,\sigma}(t) = \zeta_{L,M(L) \cup \{-1\}}(t)$$

From this the proof of the claim follows.

In the following Claim \mathbf{t}_{n+1} also denotes the cyclic permutation of the set $\{-1, 0, 1, \dots, n\}$: $\mathbf{t}_{n+1}(n) = -1$, $\mathbf{t}_{n+1}(-1) = 0$, $\mathbf{t}_{n+1}(0) = 1, \dots, \mathbf{t}_{n+1}(n-1) = n$

Claim 4. Suppose $L, L_1 \subset \{0, 1, \dots, n\}$ are such that $L = \mathbf{t}_{n+1}^k L_1$ where $k > 0$ is least with this property. Then $\zeta_L = -\mathbf{t}_{n+1}^{k+1} \zeta_{L_1}$.

Proof of the Claim. We observe that if L' is an arbitrary subset of $\{-1, 0, \dots, n\}$ such that $|L'| = n \pmod{2}$ then if we denote $L'' = \mathbf{t}_{n+1}^{-1}$ we have $\mathbf{t}_{n+1} \zeta_{L''} = \zeta_L$ for $-1 \in L$ and $\mathbf{t}_{n+1} \zeta_{L''} = \zeta_L$ for $-1 \notin L$. This is seen in the following way. In the first case $\Sigma(L'') = \Sigma(L') \pmod{2}$ and there is no

sign induced by the cyclic permutation (because an odd element “jumps” over other $|L| - 1$ odd elements). In the second case we have $\Sigma(L'') = \Sigma(L) + |L|$ and the sign of the cyclic permutation is the same as in the trivially graded case (as no odd element “jumps”), that is $(-1)^{n+1}$.

This proves the last Claim as well as the Theorem. \square

We proceed now to state and prove our main theorem after reviewing the main concepts and assumptions involved in its statement.

Assumptions and notation. We assume that we are given the following:

1. a cycle (Ω, ∇, τ) (definition 2.1), that is $\nabla : \Omega \rightarrow \Omega$ is a graded derivation such that ∇^2 is inner and the trace τ vanishes on covariant derivatives $\tau(\nabla(a)) = 0$.
2. the larger algebra $\tilde{\Omega} = \Omega \oplus \Omega X \oplus X\Omega \oplus X\Omega X$, as in equation (6), together with the canonical differential d and extension $\tilde{\tau}$ of the trace τ such that $(\tilde{\Omega}, d, \tilde{\tau})$ is an exact cycle.
3. we assume $\tau(F_{-2m-1}\Omega) = 0$ and we will use the $2m$ -cyclic cocycle φ^τ on $A = \Omega/F_{-1}\Omega$ corresponding to (Ω, ∇, τ) and $2m$. See definition 2.7.
4. a generalized θ -summable Fredholm operator in A , that is a continuous morphism $D : \mathcal{D} \rightarrow A$, with ‘index’ $\text{Ind}(D) \in K_0(A \rtimes \mathbb{Z}_2)$ defined by the equation (15).
5. we also assume that (Ω, ∇, τ) is finite summable (over \mathcal{D}), definition 4.3, that is that the morphism D extends to a morphism $\overline{D} : \overline{\Omega} \rightarrow \tilde{\Omega}$ and that $\tilde{\tau}$ vanishes on $\overline{D}([\Sigma^{(\infty)}, \overline{\Omega}])$.

The cocycle φ^τ provides us then with an $2m$ -cyclic cocycle ψ^τ on $A \rtimes \mathbb{Z}_2$ (equation (8)), and hence to a morphism, the Connes-Karoubi character, [11, 19]

$$\psi_*^\tau : K_0(A \rtimes \mathbb{Z}_2) \rightarrow \mathbb{C} \quad (33)$$

Our main problem is to compute the Connes-Karoubi character of D , more precisely $\psi_*^\tau(\text{Ind}(D))$. The main theorem will give an expression for this ‘characteristic number’ in case ψ satisfies certain further continuity conditions (finite summability).

Here $\Sigma^{(\infty)} = \cup \Sigma^{(n)}$ is an algebra of polynomial-like functions defined in equation (19), and $\overline{\Omega}$ is a certain completion of $\Omega^*(\mathcal{D})$, the universal differential graded algebra of \mathcal{D} , see equation (16).

The exponential of the supercurvature is defined by

$$\exp(-t(D + s\nabla)^2) = \overline{D}\left(\sum_{n=0}^{\infty} s^n \omega_n(t)\right)$$

Theorem. 4.12 *Using the above notation and assumptions for the cycle (Ω, ∇, τ) and the generalized θ -summable operator $D : \mathcal{D} \rightarrow A = \Omega/F_{-1}\Omega$ we have that*

$$\tau(\exp(-t(D+s\nabla)^2)) = P(st) = (st)^{2m} \psi_*^\tau(\text{Ind}(D)) + \text{'lower order terms'} \quad (34)$$

where P is a polynomial of degree $2m$ with constant coefficients and ψ^τ is the cyclic cocycle associated to τ and $2m$, assuming that τ vanishes on $F_{-2m-1}\Omega$.

All the lower terms of the polynomial P vanish if Ω is graded and ∇ is of degree 1.

Proof. We are going to use the notations explained above.

The linear map $\tilde{\tau} \circ \overline{D}$ defines a closed graded trace on $\overline{\Omega}$. Let Φ be the map defined in Lemma 2.2, with components $\Phi_n(a_0 \otimes a_1 \dots a_n) = (-1)^\mu a_0 da_1 \dots da_n$ then $\xi_n = \tilde{\tau} \circ \overline{D} \circ \Phi_n$ are cyclic cocycles on \mathcal{D} for all n and $\varphi^\tau = \xi_{2m}$ is the cyclic cocycle in the statement of the theorem.

This gives $\omega_n(t) = \Phi_n(\psi_n(t))$ using the notation of lemma 4.6, and hence

$$\tau(\exp(-t(D + s\nabla)^2)) = \sum_{n=0}^m s^{2n} \xi_{2n}(\psi_{2n}(t))$$

By theorem 4.11 we know that $\psi_{2n}(t)$ are cyclic cycles. The components $ch_{2n}(\text{Ind}(id)) \in HC_{2n}(\mathcal{D} \rtimes \mathbb{Z}_2)_{odd} \simeq HC_{2n}(\mathcal{D})_{even}$ of the Connes-Karoubi character are also cyclic cycles. Since the homology of $(\mathcal{F})_{n \geq 0}$ is singly generated, Theorem 3.6, it follows that $\psi_{2n}(t) = f_{2n}(t) ch_{2n}(\text{Ind}(D))$ in homology, for some complex functions f_{2n} . This gives

$$\xi_{2n}(\psi_{2n}(t)) = f_{2n}(t) \xi_{2n}(ch_{2n}(\text{Ind}(id)))$$

We see then that the theorem is equivalent to $f_{2n}(t) = t^{2n}$. Since these functions do not depend on any choice, we can check our computation in the simplest example when the operator D vanishes, more precisely, when $D(f) = f(0)$. We can also assume that $\Omega = \Omega^*(M, E)$, the space of smooth forms with coefficients in the smooth bundle E on a smooth compact manifold M of dimension $2n$. In this case the computation reduces to the Chern-Weyl definition of Chern classes in terms of connection and curvature. \square

The following example will spell in more detail the relation between our theorem and the Chern-Weyl construction of characteristic classes. We will use Proposition 4.4. We make the observation that the extension $\bar{\rho} : \Sigma^{(\infty)} \rightarrow \mathfrak{B}$ need not be unital and hence that the value of $\exp(-t(D + s\nabla)^2)$ at $t = 0$ is not the unit of \mathfrak{B} but rather an idempotent \bar{e} , the strong limit of $\exp(-t(D + s\nabla)^2)$ as t decreases to 0.

Consider a smooth compact manifold X and p_0, p_1 two projections in $M_n(C^\infty(X))$ and define

$$\bar{\rho}(f) = f(0)p_0 \oplus f(0)p_1 \in A = M_{2n}(C^\infty(X))$$

where A is endowed with the grading defined by the matrix $\gamma = 1 \oplus (-1)$. Define $\mathfrak{B} = \Omega = M_{2n}(\Omega^*(X))$.

Then we have $A \rtimes \mathbb{Z}_2 \simeq A \oplus A$ where the first map sends v to γ and the second one sends v to $-\gamma$. The index is given by the odd element

$$\text{Ind}(D) = ([p_0] - [p_1], -[p_0] + [p_1]) \in K_0(A) \oplus K_0(A).$$

The even super traces on the algebra Ω correspond to currents on X : $\tau(\omega) = \langle \text{Str}(\omega), c \rangle$ where $c \in \Omega^{2m}(X)'$, the dual space. The cocycles φ^τ and ψ^τ will be given by the formulae

$$\varphi^\tau(a_0, a_1, \dots, a_{2m}) = \langle \text{Str}(a_0 da_1 \dots da_{2m}), c \rangle, \quad a_0, \dots, a_{2m} \in A$$

and

$$2\psi^\tau(b_0, b_1, \dots, b_{2m}) = \langle (\text{tr} \oplus -\text{tr})(b_0 db_1 \dots db_{2m}), c \rangle, \quad b_0, \dots, b_{2m} \in A \oplus A$$

which give:

$$\begin{aligned} \psi^\tau(\text{Ind}(D)) &= \langle \text{tr}(p_0 dp_0 dp_0)^m - \text{tr}(p_1 dp_1 dp_1)^m, c \rangle / (m!) \\ &= (2\pi i)^m \langle \text{ch}(p_0) - \text{ch}(p_1), c \rangle. \end{aligned}$$

Since $\exp(-t(D + s\nabla)^2) = \exp(s^2 t^2 p_0 dp_0 dp_0)p_0 \oplus \exp(s^2 t^2 p_1 dp_1 dp_1)p_1$ we obtain

$$\tau(\exp(-t(D + s\nabla)^2)) = (st)^{2m} \langle \text{Str}(p_0 dp_0 dp_0)^m \oplus (p_1 dp_1 dp_1)^m, c \rangle / (m!)$$

and this checks with the theorem.

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